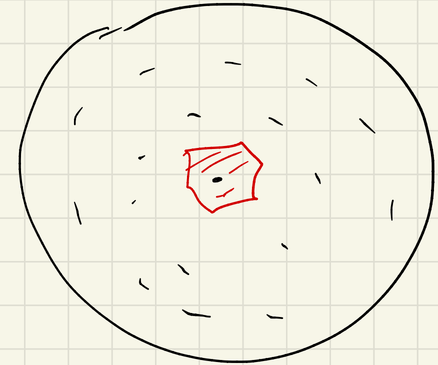
Lezione 3

M ip. completa $\iff M = \mathbb{H}^n / \Gamma$



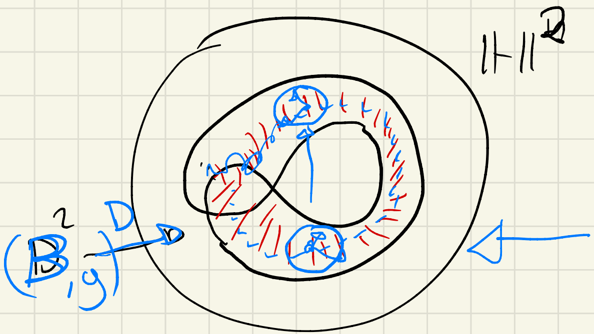
M iperbolica (anche non completa)

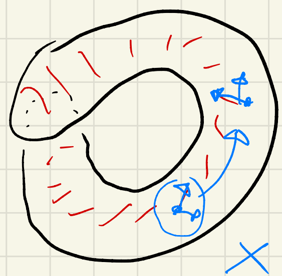
$$\begin{tikzcd} \tilde{x} \arrow[d] & \tilde{M} \arrow[r, "D"] \arrow[d, "\pi"] & \mathbb{H}^n \\ \pi_1(M,x) & M & \end{tikzcd}$$

D developing map

D isometria locale

Se M non è completa, D non è isometria
può essere non iniettiva
non suriettiva





OLONOMIA

$\rho_D : \pi_1(M) \to \mathrm{Isom}(\mathbb{H}^n)$

$\mathrm{Aut}(\pi) < \mathrm{Isom}(\tilde{M}) \xrightarrow{\rho_D} \mathrm{Isom}(\mathbb{H}^n)$

$\tilde{M} = X \xrightarrow{D} \mathbb{H}^n$ isom. locale induce $\mathrm{Isom}(X) \xrightarrow{\rho} \mathrm{Isom}(\mathbb{H}^n)$

Data $g \in \mathrm{Isom}(X)$ $\exists!$ $\rho(g) \in \mathrm{Isom}(\mathbb{H}^n)$

t.c. $D \circ g = \rho(g) \circ D$

Se M completa, $\tilde{M} \xrightarrow{D} \mathbb{H}^n$ D isometria

$\rho : \pi_1(M) \to \mathbb{H}^n$

ρ è fedele

e $\mathrm{Im}\,\rho = \Gamma$ az. lib. & p.disc.

Prop: Ogni superficie compatta euclidea è un toro
orientabile

dim:

S cpt euclidea $\Rightarrow S = \mathbb{R}^2/\Gamma$ $\quad \Gamma < \mathrm{Isom}^+(\mathbb{R}^2)$

$\Downarrow$

completa

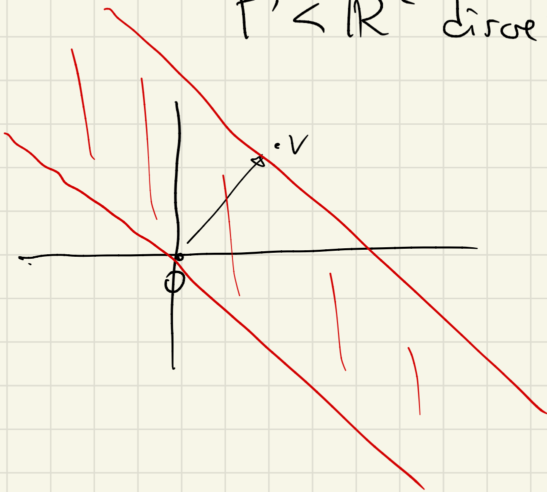
solo traslazioni

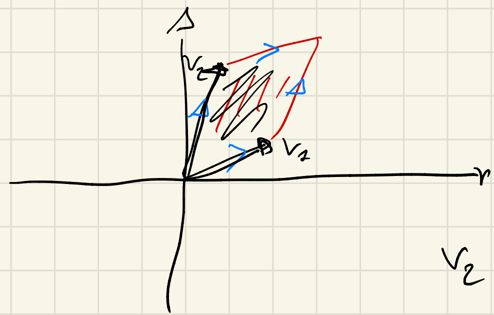
$\Rightarrow \Gamma < \mathbb{R}^2 < \mathrm{Isom}^+(\mathbb{R}^2)$

$\Gamma < \mathbb{R}^2$ discreto $\Rightarrow$ $\Gamma = \{e\}$ $\to S = \mathbb{R}^2$

$\Gamma = \mathrm{Span}(v)$ $\to S \cong S^1 \times \mathbb{R}$

$\Gamma = \mathrm{Span}(v_1, v_2)$ $\to S \cong S^1 \times S^1$





$$\mathrm{Area}(T) = |\det(v_1 \; v_2)|$$

$v_2 \rightsquigarrow v_2 + v_1$

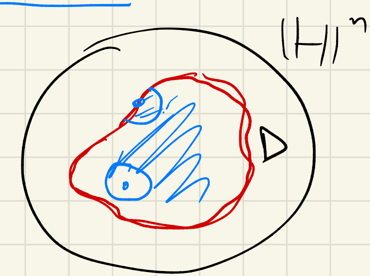
$v_1 \rightsquigarrow v_2$

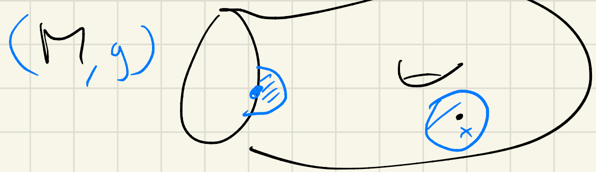
VARIETÀ IPERBOLICHE CON BORDO GEODETICO CONVESSO

Def: M varietà iperbolica con $\partial M \neq \emptyset$:= (1) $K \equiv -1$

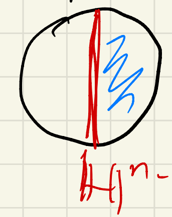
(2) loc. isom. a sottovarietà n-dim. di $\mathbb{H}^n$

(M, g)





∂M GEODETICO se geod. di ∂M sono anche geod. per M

qui è come chiedere che $D =$ 

$\Downarrow$

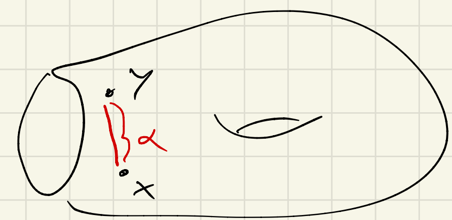
∂M è una $(n-1)$-varietà iperbolica

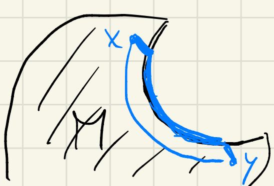
∂M CONVESSO se D è convesso

Prop: ∂M convesso $\Longleftrightarrow$ M è CONVESSA $:=$

$\forall x, y \in M$, $\forall \alpha$ che li connette

$\exists$ geod $\sim \alpha$ " " "



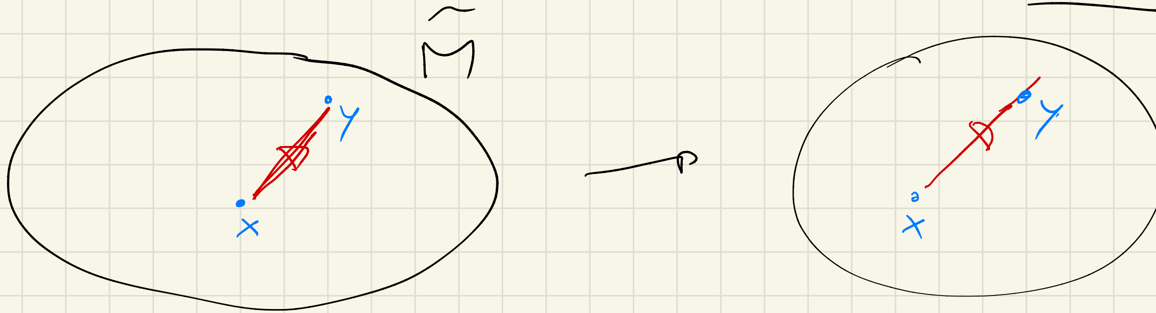


$$\ddot{x}_i + \Gamma^i_{jk}\,\dot{x}_j\,\dot{x}_k = 0$$

$M \to \tilde{M}$
conv conv

Prop: Se M è ip. convessa, allora $\tilde{M} \xrightarrow{D} \mathbb{H}^n$ è immersione INIETTIVA

dim:



Cor: $D: \tilde{M} \xrightarrow{\sim} D(\tilde{M}) \subseteq \mathbb{H}^n$
convesso dominio $\Rightarrow$ contrattile

Oss: Se ∂M geod $\Rightarrow \partial\tilde{M}$ geod. $\Rightarrow D(\tilde{M})$ ha ∂ geod.
$D(\tilde{M}) = D$

Un dominio in $\mathbb{H}^n$ con ∂ geodetico

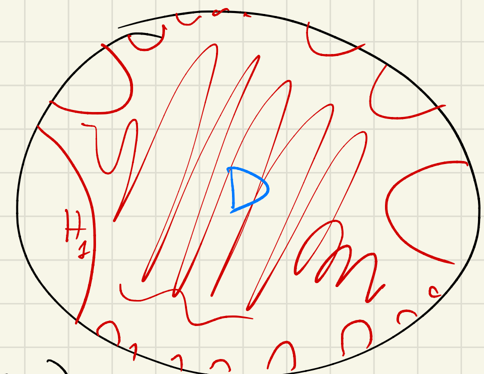
∂D è loc. come un iperpiano

Se M completa con ∂M geod $\Rightarrow \tilde{M}$ compl $\Rightarrow D(\tilde{M})$ ha ∂ geod completo

$D \subseteq \mathbb{H}^n$ ∂D completo ogni componente di ∂D è un iperpiano

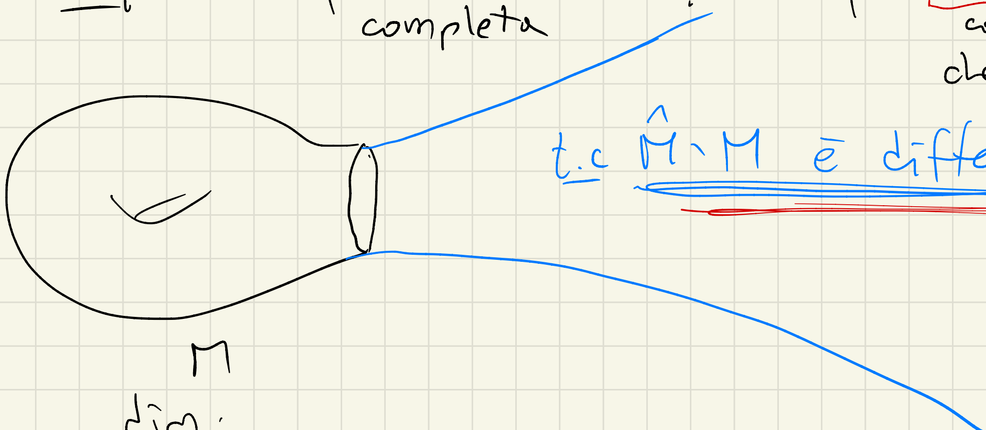
$\Rightarrow$ $\partial D = \bigcup_i H_i$ iperpiani che bordano $\overline{H}_i$

$\Rightarrow D = \bigcap \overline{H}_i$



<u>Prop</u>: M ip. convessa completa. $\exists!$ $\hat{M}$ ip. senza ∂ completa che la contiene

t.c $\hat{M} \setminus M$ è diffeom. a $\partial M \times (0, \infty)$



<u>dim</u>:

dim: compl. conv. $\tilde{M} \xrightarrow{D} \mathbb{H}^n$ isom. su img. $D(\tilde{M}) = D \subseteq \mathbb{H}^n$ conv.

$$\begin{array}{c} \tilde{M} \\ \downarrow \pi \\ M \end{array}$$

$$\rho : \underset{\pi_1(M)}{\overset{\vee\!\!|}{\mathrm{Aut}(\pi)}} \xrightarrow{\rho} \mathrm{Isom}(\mathbb{H}^n)$$

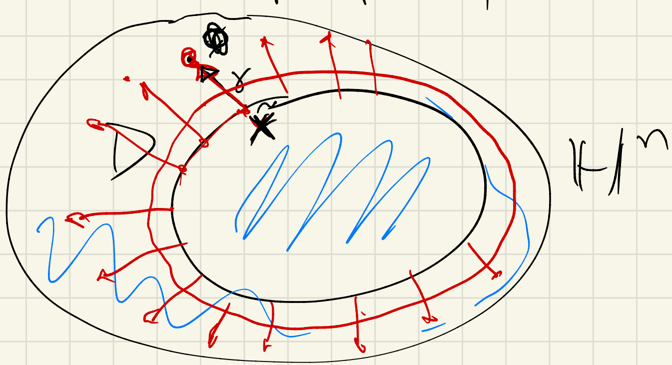
$$\Gamma = \rho(\pi_1 M)$$

Identifico $\tilde{M}$ con $D(\tilde{M})$

$$M = \tilde{M}/\mathrm{Aut}(\pi) = D(\tilde{M})/\Gamma$$

$$\Gamma < \underline{\mathrm{Isom}(D(\tilde{M}))} < \underline{\mathrm{Isom}(\mathbb{H}^n)}$$

Γ non ha pti fissi fuori da $D(\tilde{M})$.

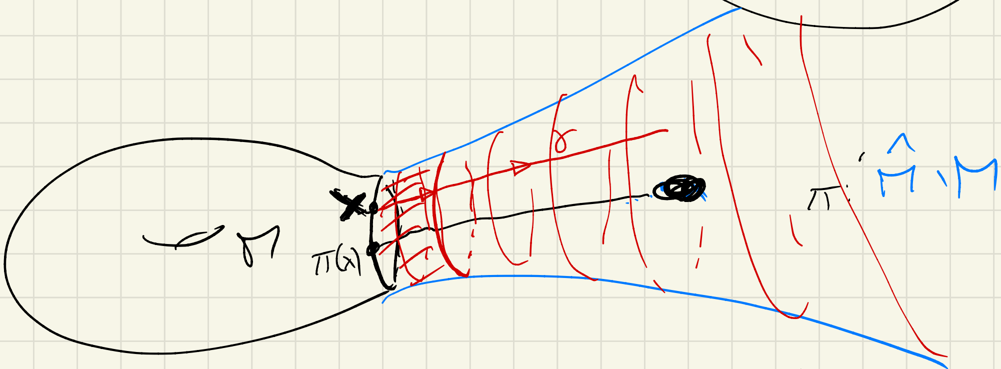


se x è pto fisso per Γ

anche y lo è

$\Rightarrow$ asurdo

$\hat{M} = \mathbb{H}^n / \Gamma$ $\qquad$ $\hat{M} \supseteq M$ $\qquad$ $M \dashrightarrow \hat{M}$



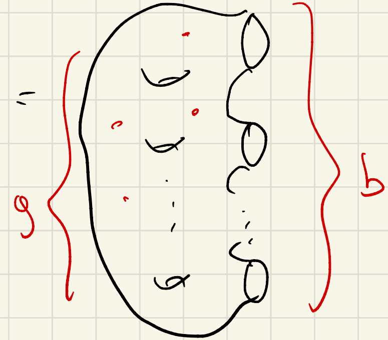
$\pi : \hat{M} \setminus M \to \partial M$

$$\psi : \partial M \times [0,\infty) \longrightarrow \hat{M} \setminus \mathrm{int}(M) \qquad \text{diffeom}$$
$$(x,t) \longmapsto \gamma(t)$$

$$\psi(\partial M \times \{t\}) = \{\text{punti di } \hat{M} \setminus M \text{ a distanza } t \text{ da } M\}$$

Strutture iperboliche su superfici

Teo (Classificazione superfici): Ogni superficie ori conn. cpt è diffeomorfa a

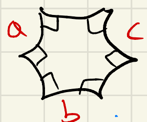
$S_{g,b} =$ 

$g, b \geq 0$

$\chi = 2 - 2g - b$

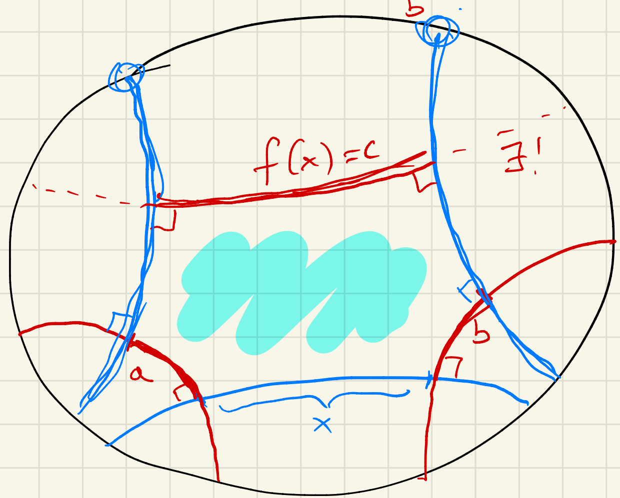
Superficie di tipo finito

$S_{g,b,p} = S_{g,b}$ con p punti interni rimossi

Prop: $\forall a,b,c > 0 \quad \exists!$  $\subseteq \mathbb{H}^2$

dim:

$\mathbb{H}^2$



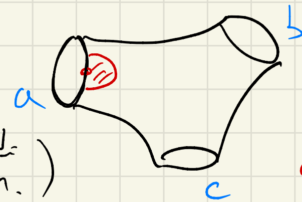
f crescente

varia da 0 a ∞

$f(x) \in (0, \infty)$

$\exists!\, x$ t.c. $f(x) = c$

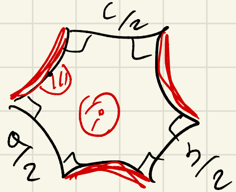
Cor: $\forall a,b,c>0 \ \exists!$ iperbolica con ∂ geod.

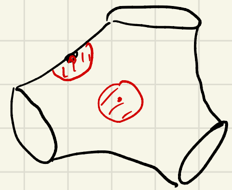


$S_{0,3}$ PANTALONI

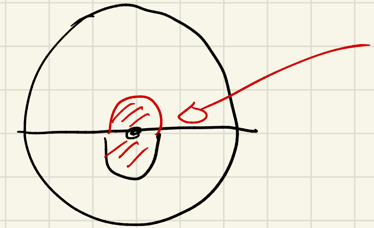
$\chi = -1$

dim: (a meno di isom.)

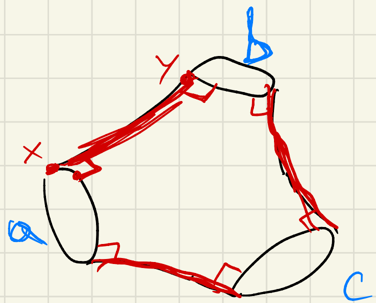


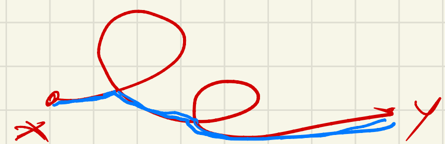


Ho strutt. di sup. ip. con ∂ geod.



Unicità

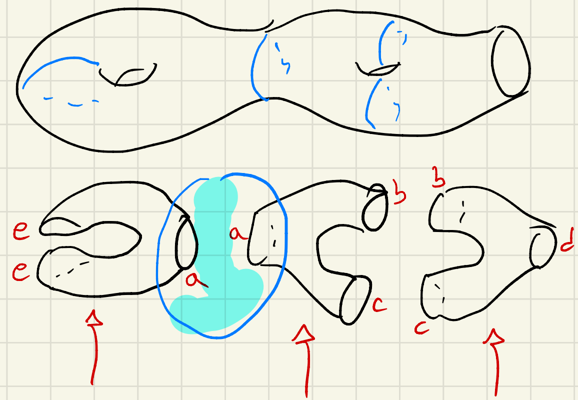




P è diviso in 2 esagoni retti isometrici $\square$

Metriche iperboliche con ∂ geod su qualsiasi $S_{g,b}$ con $\chi < 0$.
$2-2g-b$

1) $S_{g,b}$ si decompone in pantaloni



5 gradi libertà

$a, b, c, d, e > 0$

+5 gradi quali isometrie?

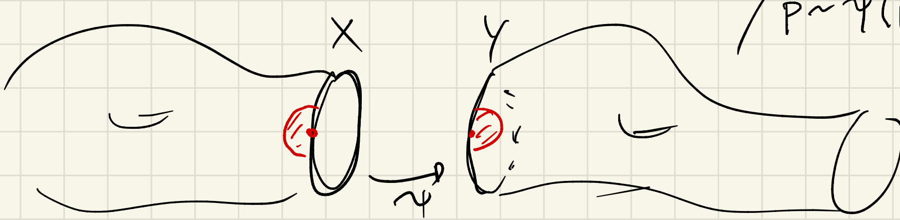
In generale: M^n, N^n ip. con bordo geod.

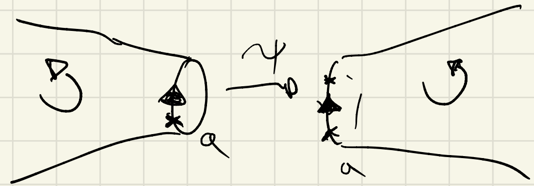
(10)

$X \subseteq \partial M \quad \partial N \supseteq Y \qquad \psi : X \to Y$ isom.

$$M \cup_\psi N = M \sqcup N \Big/ {p \sim \psi(p)}$$

ha str. di varietà ip. con ∂ geod.





esiste un S^1 insieme di possibilità